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2004 J. Phys. A: Math. Gen. 37 9439

(http://iopscience.iop.org/0305-4470/37/40/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **37** (2004) 9439–9449 PII: S0305-4470(04)82548-4

# Resonances for non-trapping time-periodic perturbations

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Received 25 June 2004, in final form 29 July 2004
Published 22 September 2004
Online at stacks.iop.org/JPhysA/37/9439
doi:10.1088/0305-4470/37/40/008

**Abstract**
For time-periodic perturbations of the wave equation in $\mathbb{R}_t \times \mathbb{R}_x^n$ given by a potential $q(t,x)$, we obtain an upper bound of the number of the resonances $\{z_j \in \mathbb{C} : |z_j| \geqslant \delta > 0\}$. We establish for $m \in N$ large enough a trace formula relating the iterations of the monodromy operator $U(mT, 0)$, $T > 0$, and the sum $\sum_j z_j^m$ of all resonances counted with their multiplicities.

PACS numbers: 02.30.Sa, 02.30.Jr, 03.65.Nk
Mathematics Subject Classification: 35P25, 35B34

## 1. Introduction

The purpose of this paper is to study the resonances of the wave equation with time-dependent potentials. Consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u + q(t,x)u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(s,x) = f_0(x), \quad u_t(s,x) = f_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the potential $q(t,x) \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$, $n \geqslant 3$, $n$ odd, satisfies the conditions:

$(H_1)$ there exists $R > 0$ such that $q(t,x) = 0$ for $|x| \geqslant R$, $\forall t \in \mathbb{R}$,
$(H_2)$ $q(t+T,x) = q(t,x)$, $\forall (t,x) \in \mathbb{R}^{n+1}$ with $T > 0$.

The solution of (1.1) is given by the propagator

$$U(t,s) : H_0 \ni (f_0, f_1) \longrightarrow U(t,s)(f_0,f_1) = (u(t,x), u_t(t,x)) \in H_0,$$

where $H_0$ is the energy space $H_0 = H_D(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ and $H_D(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H_D} = \left(\int |\nabla_x f|^2 \, \mathrm{d}x\right)^{1/2}.$$

We refer to chapter V, [13], for the properties of $U(t,s)$ and throughout the paper we will use freely the notation of [13].

Let $U_0(t) = e^{itG_0}$ be the unitary group in $H_0$ related to the Cauchy problem (1.1) with $q = 0$ and let $P_+^\rho$ (resp. $P_-^\rho$) denote the orthogonal projection on the orthogonal complement of the spaces $D_+^\rho$ (resp. $D_-^\rho$) introduced by Lax and Phillips (see [8, 13]) and defined by

$$D_\pm^\rho = \{f \in H_0 : U_0(t)f = 0 \text{ for } |x| \leqslant \pm t + \rho, \pm t \geqslant 0\}, \qquad \rho \geqslant R.$$

To define the resonances, we will use the operator

$$Z^\rho(T) = P_+^\rho U(T,0) P_-^\rho.$$

For non-trapping perturbations the spectrum of $Z^\rho(T)$ is formed by eigenvalues with finite multiplicity and for $m \in N$ large enough the operator $Z^\rho(mT)$ is compact (see [1, 5, 13]). Moreover, the eigenvalues and their multiplicity are independent of $\rho \geqslant R$ (see [5, 13]). Let $P$ denote the operator related to the problem (1.1). We define the resonances following the approach of Lax–Phillips [8] for stationary perturbations and that of Cooper–Strauss [5] for time-periodic ones.

**Definition 1.** *We say that $z \in \mathbb{C}\backslash\{0\}$ is a resonance for $P$ if $z \in \sigma_{\text{pp}} Z^\rho(T)$.*

Notice that $e^{i\sigma T}$ is an eigenvalue of $Z^\rho(T)$ if and only if there exists an outgoing solution $u(t,x)$ of the problem (1.1) with non-vanishing initial data such that $e^{-i\sigma T}u(t,x)$ is periodic with period $T$. We refer to [13] for the definition of an outgoing solution (see also [5]) as well as for the proof of the above equivalence. The second definition presents a more precise description of the existence of outgoing modes with complex frequencies known in the physical literature.

We denote by $\text{Res}\, P$ the set of resonances of $P$. In the following we write $Z(T)$, $P_\pm$ instead of $Z^\rho(T)$, $P_\pm^\rho$ if the dependence on $\rho$ is not important and we set

$$U(T) = U(T,0), \qquad Z_0(T) = P_+ U_0(T) P_-.$$

In section 3 we obtain an upper bound of the number of the resonances

$$N_\delta = \#\{z \in \text{Res}\, P : |z| \geqslant \delta\} \leqslant C_\epsilon \delta^{-\epsilon}, \qquad 0 < \epsilon \leqslant 1/2$$

which generalizes the known results for time-independent perturbations (see [11, 21, 14, 19] and the papers cited there). To the best of our knowledge this is the first upper bound for $N_\delta$ for time-periodic perturbations. Next, using the bound of the resonances, we obtain a trace formula involving the resonances. More precisely, given a function $g(z) = z^m h(z)$, holomorphic in a disk containing the resonances, we establish a trace formula involving $g(U(T))$ and the series

$$\sum_{z_j \in \text{Res}\, P} g(z_j)$$

in the spirit of trace formulae obtained for stationary perturbations in [2, 11, 15, 22, 16] (see section 4). In particular, for $h(z) = 1$ we have the following

**Theorem 1.** *Let $\chi \in C_0^\infty(B(0,r_1);[0,1])$ be such that $\chi = 1$ for $|x| \leqslant R + T$ and let*

$$\chi(U(T) - U_0(T)) = (U(T) - U_0(T))\chi = U(T) - U_0(T). \tag{1.2}$$

*Let the projectors $P_\pm$ and the number $k \in \mathbb{N}$ be fixed so that*

$$P_+ U_0(jT) P_- = 0, \quad j \geqslant k, \qquad P_\pm \chi = \chi P_\pm = \chi. \tag{1.3}$$

*Then for $m \geqslant 2k$ large enough we have*

$$\operatorname{tr}((U(kT) - U_0(kT))U(mT - 2kT)(U(kT) - U_0(kT))) = \sum_{z_j \in \operatorname{Res} P} z_j^m, \tag{1.4}$$

*where the summation is over all resonances counted with their multiplicities.*

**Remarks.**

(1) The equality (1.2) follows from the finite speed of propagation and the representation

$$U(T) = U_0(T) - \int_0^T U_0(T - s)Q(s)U(s, 0)\,\mathrm{d}s,$$

where

$$Q(s) = \begin{pmatrix} 0 & 0 \\ q(s, x) & 0 \end{pmatrix}.$$

(2) It is clear that we can choose $\psi \in C_0^\infty(\mathbb{R}^n)$ with the property $\psi = 1$ for $|x| \leqslant r_1 + kT$ so that

$$(U(kT) - U_0(kT))(1 - \psi) = 0$$

which is a consequence of

$$\chi U_0(jT)(1 - \psi) = 0, \qquad j = 0, \quad 1, \ldots, k - 1$$

(see equality (4.1)). Thus in the trace formula (1.4) on the right and on the left of $U(mT - 2kT)$ we may put the cut-off operator $\Psi = (U(kT) - U_0(kT))\psi$ acting as a multiplication operator.

**Corollary 1.** *Under the assumptions of theorem 1 the existence of a sequence $m_\nu \in \mathbb{N}$, $m_\nu \nearrow \infty$, such that*

$$|\operatorname{tr}((U(kT) - U_0(kT))U(m_\nu T)(U(kT) - U_0(kT)))| \longrightarrow \infty \qquad \text{as } m_\nu \nearrow \infty$$

*is equivalent to*

$$\operatorname{Res} P \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset.$$

The above result says that the existence of resonances $z_j, |z_j| > 1$, associated with solutions whose local energy blows up is connected to the behaviour as $m \to \infty$ of the trace of a cut-off iteration $\Psi U(mT)\Psi$. It is clear that we can choose $b > 0$ so that the projectors $P_\pm^b$ satisfy $P_\pm^b \Psi = \Psi P_\pm^b = \Psi$. Then we obtain the property

$$\operatorname{Res} P \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset \Leftrightarrow \overline{\lim}_{m\to\infty} |\operatorname{tr}(Z^b(mT))| = +\infty.$$

The result of corollary 1 seems quite natural for time-periodic perturbations. For example, the existence of intervals of instability for the Hill equation

$$y''(t) + p(t)y(t) + \lambda y(t) = 0 \tag{1.5}$$

with time-periodic $p(t)$ and $\lambda \in \mathbb{R}$ is determined by the trace

$$\operatorname{tr} M(\lambda) = y_1(T, \lambda) + y_2'(T, \lambda)$$

of the Wronskian $M(\lambda)$ given by

$$\begin{pmatrix} y(T) \\ y'(T) \end{pmatrix} = M(\lambda) \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}.$$

Here $y_1(t,\lambda)$ (resp. $y_2(t,\lambda)$) is the solution of (1.5) with $y_1(0,\lambda) = 1, y_1'(0,\lambda) = 0$ (resp. $y_2(0,\lambda) = 0, y_2'(0,\lambda) = 1$). The intervals of instability are described by the set $\{\lambda : |\mathrm{tr}\, M(\lambda)| > 2\}$ (see, for instance, [7]). Moreover, if $\lambda_0$ lies in an interval of instability, there exists an eigenvalue $\mu(\lambda_0)$, $|\mu(\lambda_0)| > 1$, of $M(\lambda_0)$, and we have

$$\lim_{m\to\infty} |\mathrm{tr}\, M^m(\lambda_0)| = \infty.$$

This phenomenon appears for the so-called parametric resonances [6] and if $\lambda = l\frac{\pi^2}{T^2}$ with suitable $l \in \mathbb{N}$ there exist unbounded solutions.

For stationary perturbations, given by a potential $V(x)$, we have always resonances and some lower bounds for the function counting the number of the resonances have been established (see [4, 3] and the references cited there). In contrast to the stationary case, for time-periodic perturbations, it is possible to construct a potential $q(t,x)$ such that the corresponding operator $P$ has no resonances $z \neq 0$. In section 5 we treat this problem.

## 2. Meromorphic continuation of the resolvent $(U(T) - z)^{-1}$

In the following $\mathcal{H}$ will denote the space $H_0$. Given a resonance $z_0 \in \mathrm{Res}\, P$, consider the projection

$$\pi_{z_0,Z} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_0} (z - Z(T))^{-1}\, \mathrm{d}z,$$

where $\gamma_0 = \{z \in \mathbb{C} : z = z_0 + \epsilon\, \mathrm{e}^{\mathrm{i}\varphi}, 0 \leqslant \varphi < 2\pi\}$ and $\epsilon > 0$ is sufficiently small. The space $\pi_{z_0,Z}(\mathcal{H})$ has a finite dimension, independent of $\rho$, and we define the multiplicity of $z_0$ as

$$m(z_0) = \mathrm{rank}\, \pi_{z_0,Z}(\mathcal{H}).$$

Let $C_0 > 0$ be a constant such that $\|U(T)\| \leqslant C_0$ and let the cut-off function $\chi$, the projectors $P_\pm$ and the integer $k \in N$ be fixed as in theorem 1.

Introduce a number $a_0 > r_1 + kT$ and let $\mathcal{H}_{R+a_0}$ be the space of the elements of $\mathcal{H}$ which vanish for $|x| \geqslant R + a_0$. Next define the space $\mathcal{H}_{\mathrm{loc}}$ as the space of functions $u$ for which $\psi u \in \mathcal{H}$ for each $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of $B(0, R + a_0)$. Then we have the following

**Proposition 1.** *The operator* $(U(T) - z)^{-1} : \mathcal{H}_{R+a_0} \longrightarrow \mathcal{H}_{\mathrm{loc}}$ *admits a meromorphic continuation from* $|z| > C_0$ *to* $\mathbb{C}$. *The poles of this continuation coincide with the resonances* Res $P$ *and the geometric multiplicities are the same. Moreover, for every* $z_0 \in \mathrm{Res}\, P$ *we have*

$$\pi_{z_0,Z}(\mathcal{H}) = \pi_{z_0,Z}\big(\mathcal{H}_{R+a_0}\big) = \pi_{z_0,U}\big(\mathcal{H}_{R+a_0}\big), \tag{2.1}$$

*where*

$$\pi_{z_0,U} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_0} (z - U(T))^{-1}\, \mathrm{d}z : \mathcal{H}_{R+a_0} \longrightarrow \mathcal{H}_{\mathrm{loc}}.$$

**Remark.** The above result is similar to proposition 3.6 in [14], where the resonances for compactly supported perturbations are defined by the method of complex scaling.

**Proof.** For $|z| > C_0$ we have $\chi(Z(T) - z)^{-1}\chi = \chi(U(T) - z)^{-1}\chi$ and the poles of $\chi(U(T) - z)^{-1}\chi$ are included in the set Res $P$. To prove the inverse, note that

$$W(T) = Z_0(T) - Z(T) = P_+(U_0(T) - U(T))P_- = \chi(U_0(T) - U(T))\chi = \chi V(T)\chi, \tag{2.2}$$

where $V(T) = U_0(T) - U(T)$. Next, we have

$$\begin{aligned}(Z(T) - z)^{-1} &= (Z(T) - z)^{-1}(Z_0(T) - Z(T))(Z_0(T) - z)^{-1} + (Z_0(T) - z)^{-1} \\ &= (Z_0(T) - z)^{-1}(Z_0(T) - Z(T))(Z(T) - z)^{-1}(Z_0(T) - Z(T))(Z_0(T) - z)^{-1} \\ &\quad + (Z_0(T) - z)^{-1}(Z_0(T) - Z(T))(Z_0(T) - z)^{-1} + (Z_0(T) - z)^{-1} \\ &= (Z_0(T) - z)^{-1}\chi V(T)\chi(U(T) - z)^{-1}\chi V(T)\chi(Z_0(T) - z)^{-1} \\ &\quad + (Z_0(T) - z)^{-1}\chi V(T)\chi(Z_0(T) - z)^{-1} + (Z_0(T) - z)^{-1}. \end{aligned} \tag{2.3}$$

The resolvent $(Z_0(T) - z)^{-1}$ is holomorphic in $\mathbb{C}\backslash\{0\}$ and (2.3) implies that the eigenvalues of $Z(T)$ are inside the poles of $\chi(U(T) - z)^{-1}\chi$. Thus the resonances coincide with the poles of the meromorphic continuation of $\chi(U(T) - z)^{-1}\chi$ and it follows immediately that the geometric multiplicities are the same.

To establish (2.1), note that according to (2.3), we have

$$\pi_{z_0,Z} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_0} (z - Z(T))^{-1}\chi V(T)\chi(Z_0(T) - z)^{-1}\,\mathrm{d}z.$$

Given $f \in \mathcal{H}$, we write

$$\chi V(T)\chi(Z_0(T) - z)^{-1} f = \sum_{j=0}^{N_0}(z - z_0)^j \chi f_{j,0} + \mathcal{O}_f((z - z_0)^{N_0+1}), \qquad z \in \gamma_0$$

and we obtain for $N_0 \gg 1$

$$\pi_{z_0,Z} f = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_0} (z - Z(T))^{-1} \sum_{j=0}^{N_0}(z - z_0)^j \chi f_{j,0}\,\mathrm{d}z.$$

On the other hand, as in the paper of Sjöstrand and Zworski [14], we get

$$\begin{aligned}&(z - z_0)^j - (Z(T) - z_0)^j \\ &\quad = (z - Z(T))[(z - z_0)^{j-1} + (z - z_0)^{j-2}(z - Z(T)) + \cdots + (z - Z(T))^{j-1}].\end{aligned}$$

For $j \geqslant 1$ we replace $(z - z_0)^j$ by $(Z(T) - z_0)^j$ and we deduce

$$\pi_{z_0,Z} f = \pi_{z_0,Z}\left(\sum_{j=0}^{N_0}(Z(T) - z_0)^j(\chi f_{j,0})\right).$$

Next we exploit the equality

$$Z(jT) - Z_0(jT) = -\sum_{\nu=0}^{j-1} Z_0(\nu T)(Z_0(T) - Z(T))Z((j - \nu - 1)T).$$

Observing that $Z_0(\nu T) = 0$ for $\nu \geqslant k$, we deduce

$$Z(jT)\chi = Z_0(jT)\chi - \sum_{\nu=0}^{k-1} Z_0(\nu T)\chi V(T)\chi Z((j - \nu - 1)T)\chi.$$

This implies

$$Z(jT)\chi = P_+\Phi, \qquad \forall j \in \mathbb{N},$$

where $\Phi \in C_0^\infty(B(0, r_1 + kT); [0, 1])$ is such that $(1 - \Phi)U_0(jT)\chi = 0$ for $0 \leqslant j \leqslant k - 1$. Since

$$\pi_{z_0,Z} P_+\Phi = P_+\pi_{z_0,Z}\Phi = \pi_{z_0,Z}\Phi,$$

we conclude that

$$\pi_{z_0,Z}(\mathcal{H}) = \pi_{z_0,Z}(\Phi\mathcal{H}) \subset \pi_{z_0,Z}\big(\mathcal{H}_{R+a_0}\big).$$

Finally, if $P_-^\rho \Phi = \Phi$ we have

$$\pi_{z_0,Z}(\mathcal{H}) = \pi_{z_0,U}(\Phi\mathcal{H}) + \frac{1}{2\pi\mathrm{i}}\big(1 - P_+^\rho\big)\int_{\gamma_0}(z - U(T))^{-1}\Phi\,\mathrm{d}z : \mathcal{H}_{R+a_0} \longrightarrow \mathcal{H}_{\mathrm{loc}}.$$

The term involving $\big(1 - P_+^\rho\big)$ is independent of the choice of $P_+^\rho$, provided $P_-^\rho\Phi = \Phi$ and it vanishes on every compact set. This completes the proof. □

## 3. Upper bound of the number of resonances

In this section, we give a upper bound of the number of resonances lying in the disc

$$\{z \in \mathbb{C} : |z| \geqslant \delta\}, \qquad \delta > 0.$$

We will prove the following

**Theorem 2.** *Suppose the assumptions* $(H_1), (H_2)$ *are fulfilled. Then the number of the resonances* $z \in \operatorname{Res} P$, $|z| > 1$*, is finite and for each* $\varepsilon > 0$ *there exists a constant* $C_\epsilon > 0$ *such that for every* $0 < \delta \leqslant 1$ *we have*

$$\#\{z \in \operatorname{Res} P : |z| \geqslant \delta\} \leqslant C_\epsilon \delta^{-\epsilon}. \tag{3.1}$$

**Remarks.**

(1) For stationary potentials this result has been obtained by Melrose [11] (see the estimate (44)).

(2) The above bound is natural for independent on time perturbations. Indeed, in this case, Melrose [11], Zworski [21], Sjöstrand and Zworski [14] and Vodev [19] have proved that

$$\#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : |\sigma| \leqslant r\} \leqslant C r^n. \tag{3.2}$$

Moreover, if $P$ is non-trapping, Vainberg [17] in the classical case and Martinez [10] in the semi-classical framework have shown that for each $N \in \mathbb{N}$ we have

$$\#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : |\operatorname{Im}\sigma| \leqslant N\ln(|\sigma|)\} < \infty. \tag{3.3}$$

This implies

$$\begin{aligned}\#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : |\operatorname{Im}\sigma| \leqslant r\} &\leqslant \#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : N\ln(|\sigma|) \leqslant |\operatorname{Im}\sigma| \leqslant r\} + C_N \\ &\leqslant \#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : |\sigma| \leqslant \mathrm{e}^{r/N}\} + C_N \leqslant C_N' \,\mathrm{e}^{rn/N}.\end{aligned} \tag{3.4}$$

Now, fixing $T > 0$ and setting $z = \mathrm{e}^{\mathrm{i}\sigma T}$, we obtain the estimate (3.1) with $\epsilon = \frac{n}{TN}$.

**Proof.** We will exploit the method developed by Melrose [11, 12] for perturbations independent of time (see also Zworski [21] and Vodev [19]). To prove the theorem, it is sufficient to show that there exists $N \in \mathbb{N}$ such that for each $\varepsilon > 0$, the eigenvalues of the operator $Z(NT)$ satisfy for all $0 < \delta \leqslant 1$ the estimate

$$\#\{z \in \mathbb{C} : z \in \sigma_{\mathrm{pp}}(Z(NT)), |z| \geqslant \delta\} \leqslant C_\epsilon \delta^{-\varepsilon}. \tag{3.5}$$

Given a compact operator $S$, we denote by $\mu_j(S)$, $j = 1, 2, \ldots$, the characteristic values of $S$ which form a non-increasing sequence of the eigenvalues of $(S^*S)^{1/2}$ counted with their multiplicity. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $k \in \mathbb{N}$ be fixed as in theorem 1 so that $Z_0(kT) = 0$. For $M \in \mathbb{N}$, we have

$$\begin{aligned}
Z((2k+M)T) &= Z(kT)Z(MT)Z(kT) \\
&= (Z(kT) - Z_0(kT))Z(MT)(Z(kT) - Z_0(kT)) \\
&= P_+(U(kT) - U_0(kT))U(MT)(U(kT) - U_0(kT))P_- \\
&= P_+(U(kT) - U_0(kT))\chi U(MT)\chi(U(kT) - U_0(kT))P_-. \qquad (3.6)
\end{aligned}$$

Since the perturbation of $P$ is given by a potential, the results for the propagation of singularities imply that the operator $\chi U(MT)\chi$ is regularizing for $M \in \mathbb{N}$ large enough (see [5, 1, 13, 18]). Let $\Omega \subset\subset \mathbb{R}^{2n}$ be an open hypercube, with $\operatorname{supp}\chi \subset \Omega$, and let $\Delta_\Omega$ be the Laplacian in $\Omega$ with Dirichlet boundary condition. It is well known (see for instance, [21, 19]) that for all $m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$\mu_j((I - \Delta_\Omega)^{-m}) \leqslant C_m j^{-2m/n}, \qquad \forall j \in \mathbb{N}.$$

Consequently, using (3.6) and the inequalities

$$\mu_j(AB) \leqslant \mu_j(A)\|B\|, \qquad \mu_j(AB) \leqslant \mu_j(B)\|A\|,$$

we get, for $m \in \mathbb{N}$,

$$\begin{aligned}
\mu_j(Z((2k+M)T)) &\leqslant C\mu_j(\chi U(MT)\chi) \\
&\leqslant C\mu_j((I - \Delta_\Omega)^{-m}(I - \Delta_\Omega)^m \chi U(MT)\chi) \\
&\leqslant C\mu_j((I - \Delta_\Omega)^{-m})\|(I - \Delta_\Omega)^m \chi U(MT)\chi\| \\
&\leqslant C_m j^{-2m/n} \qquad (3.7)
\end{aligned}$$

with a new constant $C_m > 0$.

We choose $N = 2k + M$, $2m > n$ and we order the eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_p, \ldots$$

of $Z(NT)$ counted with their multiplicities by decreasing modulus. Then

$$|\lambda_p|^p \leqslant \prod_{j=1}^p |\lambda_j| \leqslant \prod_{j=1}^p \mu_j(Z(NT)) \leqslant (C_k)^p (p!)^{-k},$$

where $k \in \mathbb{N}$ can be taken as large as we wish. Thus with a constant $C'_k$, we get

$$|\lambda_p| \leqslant C_k (p!)^{-\frac{k}{p}} \leqslant C'_k p^{-k}.$$

Now for the eigenvalues $\lambda_1, \ldots, \lambda_p$ with modulus greater than $\delta > 0$ we deduce

$$p \leqslant C''_k \delta^{-\frac{1}{k}}$$

and taking $k = \frac{1}{\epsilon}$, we complete the proof. □

## 4. Trace formula

In this section we prove theorem 1. Recall that $\chi \in C_0^\infty(\mathbb{R}^n)$, the projectors $P_\pm$ and $k \in \mathbb{N}$ are fixed so that (1.2) and (1.3) hold. First notice that

$$\begin{aligned}
U(kT) - U_0(kT) &= \sum_{j=0}^{k-1} U(jT)(U(T) - U_0(T))U_0((k-j-1)T) \\
&= P_-(U(kT) - U_0(kT)) = (U(kT) - U_0(kT))P_+. \qquad (4.1)
\end{aligned}$$

The second and the third equalities follow from the fact that

$$(I - P_-)U(jT)\chi = \chi U_0(jT)(I - P_+) = 0, \qquad j = 0, \ldots, k-1.$$

The operator

$$P_+U(mT - 2kT)P_-$$

is trace class for $m$ sufficiently large and the cyclicity of the trace implies

$$\begin{aligned}
&\mathrm{tr}((U(kT) - U_0(kT))U(mT - 2kT)(U(kT) - U_0(kT))) \\
&\quad = \mathrm{tr}(P_-(U(kT) - U_0(kT))P_+U(mT - 2kT)P_-(U(kT) - U_0(kT))P_+) \\
&\quad = \mathrm{tr}(P_+(U(kT) - U_0(kT))P_-P_+U(mT - 2kT)P_-P_+(U(kT) - U_0(kT))P_-) \\
&\quad = \mathrm{tr}(P_+U(kT)P_-P_+U(mT - 2kT)P_-P_+(U(kT)P_-) \\
&\quad = \mathrm{tr}(P_+U(mT)P_-) = \mathrm{tr}(Z(mT)).
\end{aligned}$$

Applying Lidsii theorem for the trace of $Z(mT)$, we complete the proof since by theorem 2 we have

$$\left|\sum_j z_j^m\right| \leqslant \sum_{p=1}^{\infty} \sum_{\frac{C}{p+1} < |z_j| \leqslant \frac{C}{p}} \left|z_j^m\right| \leqslant C_\epsilon \sum_{p=1}^{\infty} \left(\frac{C}{p}\right)^{m-\epsilon} \leqslant C_m, \qquad 0 < \epsilon \leqslant 1/2, \quad m \geqslant 2.$$

It is clear that corollary 1 follows from the following

**Lemma 1.** *Let*

$$A_m = \sum_{|z_j| \leqslant 1} z_j^m, \qquad B_m = \sum_{|z_j| > 1} z_j^m, \qquad m \in \mathbb{N}.$$

*Then*

$$|A_m| \leqslant C_0, \qquad \forall m \geqslant 1 + \epsilon_0 > 1.$$

*Moreover, if* $\{z \in \mathrm{Res}\, P : |z| > 1\} \neq \emptyset$*, then there exists a sequence* $m_\nu \nearrow \infty, m_\nu \in \mathbb{N}$*, such that*

$$\lim_{m_\nu \to \infty} \left|B_{m_\nu}\right| = \infty.$$

**Proof.** Let $m - \epsilon > 1, \epsilon > 0$. Using the estimate

$$\#\{z_j \in \mathrm{Res}\, P : |z_j| \geqslant \delta\} \leqslant C_\epsilon \delta^{-\epsilon},$$

we obtain

$$|A_m| \leqslant \sum_{k=1}^{\infty} \sum_{\frac{1}{k+1} < |z_j| \leqslant \frac{1}{k}} |z_j|^m \leqslant C_\epsilon \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^m \left(\frac{1}{k+1}\right)^{-\epsilon} \leqslant C'_\epsilon.$$

To deal with the sum $B_m$, introduce

$$\mu = \max\{|z_j| : z_j \in \mathrm{Res}\, P, |z_j| > 1\}.$$

Since we have a finite number of resonances $z_j$ with $|z_j| > 1$, let

$$z_j = \mu\, \mathrm{e}^{\mathrm{i}\varphi_j}, \quad j = 1, \ldots, p, \quad \varphi_\nu \neq \varphi_j \ (\mathrm{mod}\, 2\pi), \quad \nu \neq j.$$

It is sufficient to show that for a suitable sequence $m_\nu \nearrow \infty$ we have

$$\lim_{m_\nu \to \infty} \left|\sum_{j=1}^{p} c_j\, \mathrm{e}^{\mathrm{i}m_\nu\varphi_j}\right| \geqslant \epsilon_0 > 0,$$

where $c_j \in \mathbb{N}$ is the multiplicity of the resonance $z_j$, $j = 1, \ldots, p$.

Put $a_j = \mathrm{e}^{\mathrm{i}\varphi_j}$, $j = 1, \ldots, p$ and assume that

$$\lim_{m\to\infty} \sum_{j=1}^{p} c_j a_j^m = 0$$

for some integers $c_j \in \mathbb{N}$, $j = 1, \ldots, p$. Obviously,

$$\sum_{j=1}^{p} a_j^q c_j a_j^m \longrightarrow_{m\to\infty} 0 \qquad \text{for } q = 0, 1, \ldots, p-1.$$

This implies

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_p \\ \ldots & \ldots & \ldots & \\ a_1^{p-1} & a_2^{p-2} & \ldots & a_p^{p-1} \end{pmatrix} \begin{pmatrix} c_1 a_1^m \\ c_2 a_2^m \\ \ldots \\ c_p a_p^m \end{pmatrix} \longrightarrow 0$$

and we deduce that $\left(c_1 a_1^m, \ldots, c_p a_p^m\right) \longrightarrow 0$ which is a contradiction. Thus there exists a sequence $m_\nu \nearrow \infty$ such that

$$\sum_{j=1}^{p} c_j a_j^{m_\nu} \longrightarrow \beta \neq 0 \qquad \text{as } m_\nu \to \infty$$

and this completes the proof. □

Finally, we may establish a trace formula for the operator

$$g(U(T)) = U((m-2k)T) \sum_{j=0}^{\infty} b_j U(jT),$$

where the series $h(z) = \sum_{j=0}^{\infty} b_j z^j$ has a radius of convergence $R_0 > \|U(T)\|$ and $m \in \mathbb{N}$ is chosen so that $Z((m-2k)T)$ is a trace class. First note that

$$\left\| Z((m-2k)T) \sum_{j=p}^{p+q} b_j Z(jT) \right\|_{\mathrm{tr}} \leqslant \|Z((m-2k)T)\|_{\mathrm{tr}} \sum_{j=p}^{p+q} |b_j| \|Z(T)\|^j \leqslant \epsilon$$

for $p, q \geqslant N(\epsilon)$. Since the space of trace class operators is complete in trace norm, we deduce that $g(Z(T))$ is trace class and this yields

$$\mathrm{tr}\left( Z((m-2k)T) \sum_{j=0}^{N} b_j Z(jT) \right) \longrightarrow \mathrm{tr}(g(Z(T))) \qquad \text{as } N \to \infty.$$

Next, the operator

$$(U(kT) - U_0(kT)) U(mT - 2kT) \sum_{j=0}^{N} b_j U(jT)(U(kT) - U_0(kT))$$

converges in the operator norm to $(U(kT) - U_0(kT))g(U(T))(U(kT) - U_0(kT))$ and

$$\mathrm{tr}\left( (U(kT) - U_0(kT)) U(mT - 2kT) \sum_{j=0}^{N} b_j U(jT)(U(kT) - U_0(kT)) \right) \longrightarrow \mathrm{tr}\, g(Z(T)).$$

Applying the result of Gohberg and Krein (see chapter 6 in [9]), we obtain the following

**Theorem 3.** *Let* $g(z) = z^{m-2k}h(z) = z^{m-2k}\sum_{j=0}^{\infty} b_j z^j$ *be a function such that the series* $\sum_{j=0}^{\infty} b_j z^j$ *has in* $\mathbb{C}$ *a radius of convergence* $R_0 > \|U(T)\|$ *and let* $m, k$ *be chosen as in theorem 1. Then*

$$\mathrm{tr}((U(kT) - U_0(kT))g(U(T))(U(kT) - U_0(kT))) = \sum_{z_j \in \mathrm{Res}\, P} g(z_j).$$

## 5. Example

In this section we construct a potential $q(t, x)$ such that $Z(T) = 0$ which implies that we have no resonances $z \in \mathrm{Res}\, P\backslash\{0\}$ . Assume that $T = t_1 + t_0, t_1 > 0, t_0 > 0$. We choose a potential $q(t, x) \neq 0$ satisfying the assumptions $(H_1)$, $(H_2)$ such that

$$q(t, x) = 0 \quad \text{for } 0 < t_0 \leqslant t \leqslant T, \qquad \forall x. \tag{5.1}$$

Moreover, the support of $q(t, x)$ with respect to $x$ is independent of $t_0, t_1$. We obtain

$$U(T, 0) = U(t_1 + t_0, 0) = U(T, t_0)U(t_0, 0) = U_0(t_1)U(t_0, 0).$$

Here we have used the fact that (5.1) implies $U(T, t_0) = U_0(T - t_0) = U_0(t_1)$. We fix the projectors $P_+$, $P_-$, independently of $t_1$, so that $P_\pm Q(s) = Q(s)$. Next we choose the time $t_1$ large enough so that

$$P_+U_0(t_1)P_- = 0.$$

This implies

$$Z(T) = P_+U(T, 0)P_- = P_+U_0(t_1)P_-U(t_0, 0)P_- = 0,$$

since $(I - P_-)U(t_0, 0)P_- = 0$.

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